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1990 J. Phys. A: Math. Gen. 23 133

(http://iopscience.iop.org/0305-4470/23/2/010)

J. Phys. A: Math. Gen. **23** (1990) 133–146. Printed in the UK

Solvable potentials generated by SL(2,*R*)

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Received 8 August 1989

Abstract. Canonical transformations are used to build realisations of SL(2, *R*) in terms of the basic quantum mechanical operators Q and P. The results are used to construct solvable potentials in the framework of one-dimensional quantum mechanics.

1. Introduction

The purpose of this paper is to sketch a systematic search for exactly solvable potentials in one-dimensional quantum mechanics. Our approach will be based on Lie group theory. The question we ask is whether it is possible to write a Schrödinger equation

$$\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}q^2}+V(q)-E\right)\Psi(q)=0 \tag{1.1}$$

into the form

$$(h(X_1,\ldots,X_n)-\varepsilon)|\Psi\rangle=0 \tag{1.2}$$

where h is a function of Lie group generators $X_1,\ldots,X_n$. Many well known problems can be written in this way. Either they are the Casimir operator (Sukumar 1986) or a linear combination of generators (Wybourne 1974). We want to be more general. Presented in such a manner, the problem is of course too wide; we need to introduce restrictions. These will be of two types. We can make restrictions on the function h in (1.2). In the following we will use at most quadratic functions of the generators $X_1,\ldots,X_n$. Another possible restriction concerns the choice of the Lie group or more precisely of its Lie algebra. In this paper we will use the special linear group SL(2, *R*), which is isomorphic to SU(1, 1) and is well known to have finite- and infinite-dimensional irreducible representations. In order to make the problems (1.1) and (1.2) equivalent, it is necessary to realise the generators of SL(2, *R*) as functions of the canonical operators Q and P associated respectively with the position and impulsion of the particle. Here also we make a restriction, we only study the special cases where the X_i are linear or quadratic functions of the operator P, the reason being that the Schrödinger equation is quadratic in P. Note that in the Schrödinger picture our generators will be differential operators of the first or second order. The case of linear generators in P and a quadratic function h has been already discussed by Turbiner (1988). We will use nonlinear canonical transformations (Mello and Moshinsky 1975) to reduce the problem to a simpler form. In the section 2 we will give all details about the canonical transformations we will use. Section 3 is devoted to the construction of the generators of SL$(2, R)$ with the above-mentioned restrictions. In section 4 we will be concerned with the study of an algebraic model like (1.2) and its connection with the usual problem (1.1). The next two sections are devoted to examples. Finally in the last section we give some conclusions and suggest some extensions of this work.

2. Canonical transformations

The canonical transformations we will use here are simple ones. They are related to a change of variables and functions. Our two basic operators Q and P are given respectively by q and $-\mathrm{i}\,\mathrm{d}/\mathrm{d}q$ in the Schrödinger picture. The change of variable $\overline{q} = f(q)$ where f is a given function of q implies that the derivative is transformed into

$$\frac{\mathrm{d}}{\mathrm{d}\overline{q}} = \frac{1}{f'(q)}\frac{\mathrm{d}}{\mathrm{d}q}. \tag{2.1}$$

In operator notation this last expression becomes

$$\overline{P} = \frac{1}{f'(Q)}P. \tag{2.2}$$

Combined with $\overline{Q} = f(Q)$, this defines a canonical transformation. It is indeed very easy to verify that the new operators $\overline{Q}$ and $\overline{P}$ satisfy the same commutation relations as Q and P, $[\overline{Q}, \overline{P}] = \mathrm{i}$. Now we want to make our canonical transformation more general by including changes of functions. Again we go to the Schrödinger picture. A relation like

$$|\Psi\rangle = A|\Phi\rangle \tag{2.3}$$

where A is any linear operator, becomes

$$\Psi(q) = A\Phi(q). \tag{2.4}$$

Making the change of functions

$$\Psi(q) = \phi(q)\overline{\Psi}(q) \qquad \Phi(q) = \phi(q)\overline{\Phi}(q) \tag{2.5}$$

we get

$$\overline{\Psi}(q) = \left(\frac{1}{\phi(q)}A\phi(q)\right)\overline{\Phi}(q) \tag{2.6}$$

and the operator A is transformed into $\overline{A}$ given by

$$\overline{A} = \frac{1}{\phi(Q)}A\phi(Q). \tag{2.7}$$

In particular, the basic operators Q and P become

$$\begin{aligned}\overline{Q} &= \frac{1}{\phi(Q)} Q\phi(Q) = Q \\ \overline{P} &= \frac{1}{\phi(Q)} P\phi(Q) = P - \mathrm{i}\frac{\phi'(Q)}{\phi(Q)}.\end{aligned} \tag{2.8}$$

Combining (2.2) with (2.8), we finally obtain the canonical transformation we will use in the next section

$$\begin{aligned}\overline{Q} &= UQU^{-1} = f(Q) \\ \overline{P} &= UPU^{-1} = \frac{1}{f'(Q)} P + g(Q)\end{aligned} \tag{2.9}$$

where $f(Q)$ and $g(Q)$ are arbitrary functions and U denotes the operator that represents this canonical transformation in the Hilbert space.

3. Realisations of SL(2,R)

The realisations of SL(2, R) we want to discuss in this paper are of two types. Either the generators are linear or they are quadratic in P, which means that in the Schrödinger picture they appear as differential operators of the first or the second order. In both cases we will use the results of the preceding section in order to simplify the construction of the generators. The commutation rules of the generators of SL(2, R) are written as

$$[T_0, T_\pm] = \pm T_\pm \qquad [T_+, T_-] = -2T_0. \tag{3.1}$$

We now examine in detail the two special cases we are interested in.

3.1. Linear case

Let us show that the most general expression for the generators is obtained by applying the canonical transformation (2.9) to a particular case that we denote by $\overline{T}_i$:

$$T_i = U\overline{T}_i U^{-1}. \tag{3.2}$$

These special $\overline{T}_i$ are choosen so that $\overline{T}_0$ is equal to

$$\overline{T}_0 = \mathrm{i}QP - \lambda. \tag{3.3}$$

Note that this choice is completely arbitrary. Using (2.9) this gives

$$T_0 = \mathrm{i}\frac{f(Q)}{f'(Q)} P + (\mathrm{i}f(Q)g(Q) - \lambda). \tag{3.4}$$

As $f(Q)$ and $g(Q)$ are arbitrary functions of Q, so also are the coefficients of the linear expression in P (3.4). This proves the validity of (3.2). We have now to build $\overline{T}_+$ and $\overline{T}_-$ by imposing commutation rules analogous to (3.1). A very obvious calculation yields

$$\overline{T}_+ = \mathrm{i}Q^2P - 2\lambda Q \qquad \overline{T}_- = \mathrm{i}P. \tag{3.5}$$

This is one of the usual realisations (Miller 1968) of SL$(2, R)$. The generators $\overline{T}_+$ and $\overline{T}_-$ can also be written in a more convenient form

$$\overline{T}_+ = Q(\overline{T}_0 - \lambda) \qquad \overline{T}_- = \frac{1}{Q}(\overline{T}_0 + \lambda). \tag{3.6}$$

If we now go to the general expression by using the canonical transformation (2.9) we obtain

$$T_0 = \mathrm{i}\frac{f(Q)}{f'(Q)}P + F(Q) \qquad T_+ = f(Q)(T_0 - \lambda) \qquad T_- = \frac{1}{f(Q)}(T_0 + \lambda) \tag{3.7}$$

where $F(Q) = \mathrm{i}f(Q)g(Q) - \lambda$. The most general linear case depends thus on two arbitrary functions $f(Q)$ and $g(Q)$, or any combination of them. The constant λ is also arbitrary and is related to the Casimir operator

$$C = T_+T_- + T_0 - T_0^2. \tag{3.8}$$

Replacing the operators by their expression (3.7), it is very easy to show that the Casimir operator C is a multiple of the unity operator

$$C = -\lambda(\lambda + 1)I. \tag{3.9}$$

3.2. Quadratic case

We proceed in a similar way as for the linear case. The general expression for a generator will be taken to be

$$T_i = \alpha_i(Q)P^2 + \beta_i(Q)P + \gamma_i(Q). \tag{3.10}$$

Let us show that for one of the generators, T_0 for instance, its general form can be obtained by applying (2.9) to a particular form. We choose

$$\overline{T}_0 = P^2 + V(Q) \tag{3.11}$$

where V(Q) is an arbitrary function of Q. Compared to (3.10) we thus impose $\alpha_0(Q) = 1$ and $\beta_0(Q) = 0$. This is possible because the canonical transformation (2.9) contains two arbitrary functions. We obtain indeed

$$\begin{aligned} T_0 &= U(P^2 + V(Q))U^{-1} \\ &= \left(\frac{1}{f'(Q)}P + g(Q))^2 + V(f(Q)\right) \\ &= \frac{1}{[f'(Q)]^2}P^2 + \left(2\frac{g(Q)}{f'(Q)} + \mathrm{i}\frac{f''(Q)}{[f'(Q)]^3}\right)P \\ &\quad + \left(g^2(Q) - \mathrm{i}\frac{g'(Q)}{f'(Q)} + V(f(Q))\right) \end{aligned} \tag{3.12}$$

from which we deduce the system

$$
\begin{aligned}
\alpha_0(Q) &= \frac{1}{[f'(Q)]^2} \\
\beta_0(Q) &= 2\frac{g(Q)}{f'(Q)} + \mathrm{i}\frac{f''(Q)}{[f'(Q)]^3} \\
\gamma_0(Q) &= g^2(Q) - \mathrm{i}\frac{g'(Q)}{f'(Q)} + V(f(Q))
\end{aligned}
\tag{3.13}
$$

giving the link between the functions $(\alpha_0(Q), \beta_0(Q), \gamma_0(Q))$ and the functions $(f(Q), g(Q), V(Q))$. If we now take a general form similar to (3.10) for $\overline{T}_+$ and $\overline{T}_-$

$$
\overline{T}_\pm = \overline{\alpha}_\pm(Q)P^2 + \overline{\beta}_\pm(Q)P + \overline{\gamma}_\pm(Q)
\tag{3.14}
$$

and combine this with $\overline{T}_0$ given by (3.11) in order to obey the commutation relations (3.1), we obtain (Lánik 1967) up to a translation

$$
\overline{T}_0 = P^2 + \tfrac{1}{16}Q^2 + \lambda Q^{-2} \qquad \overline{T}_\pm = P^2 \pm \mathrm{i}\tfrac{1}{2}QP \pm \tfrac{1}{4} - \tfrac{1}{16}Q^2 + \lambda Q^{-2}.
\tag{3.15}
$$

For the most general quadratic generators of SL(2, R) this yields

$$
\begin{aligned}
T_0 &= U\overline{T}_0U^{-1} \\
&= \frac{1}{[f'(Q)]^2}P^2 + \left(2\frac{g(Q)}{f'(Q)} + \mathrm{i}\frac{f''(Q)}{[f'(Q)]^3}\right)P \\
&\qquad + \left(g^2(Q) - \mathrm{i}\frac{g'(Q)}{f'(Q)} + \frac{1}{16}f^2(Q) + \lambda f^{-2}(Q)\right) \\
T_\pm &= U\overline{T}_\pm U^{-1} \\
&= \frac{1}{[f'(Q)]^2}P^2 + \left(2\frac{g(Q)}{f'(Q)} + \mathrm{i}\frac{f''(Q)}{[f'(Q)]^3} \pm \frac{\mathrm{i}}{2}\frac{f(Q)}{f'(Q)}\right)P \\
&\qquad + \left(g^2(Q) - \mathrm{i}\frac{g'(Q)}{f'(Q)} \pm \frac{\mathrm{i}}{2}f(Q)g(Q) \pm \frac{1}{4} - \frac{1}{16}f^2(Q) + \lambda f^{-2}(Q)\right).
\end{aligned}
\tag{3.16}
$$

Again there is an arbitrary constant λ in the results. As in the linear case, it is associated with the Casimir operator (3.8). A very simple calculation shows that

$$
C = \tfrac{1}{16}(3 - 4\lambda)I.
\tag{3.17}
$$

Note that the case $\lambda = 0$ is a special well known case corresponding to the one-dimensional harmonic oscillator. In fact it is possible to define boson creation and annihilation operators

$$
\overline{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{1}{2}Q - 2\mathrm{i}P\right) \qquad \overline{a} = \frac{1}{\sqrt{2}}\left(\frac{1}{2}Q + 2\mathrm{i}P\right).
\tag{3.18}
$$

They obey the usual commutation rules $[\overline{a}, \overline{a}^\dagger] = 1$ and allow us to write the generators $\overline{T}_i$ as

$$
\overline{T}_0 = \tfrac{1}{2}(\overline{a}^\dagger\overline{a} + \tfrac{1}{2}) \qquad \overline{T}_+ = -\tfrac{1}{2}\overline{a}^\dagger\overline{a}^\dagger \qquad \overline{T}_- = -\tfrac{1}{2}\overline{a}\overline{a}.
\tag{3.19}
$$

The method used here shows that this decomposition of the generators of $SL(2, R)$ is still valid for the general expression (3.16) provided that λ is equal to zero. The expressions are similar to (3.19) except that we have to drop the bars over the operators. When λ is different from zero this is no longer true. Nevertheless $\overline{T}_0$ can be written in a similar form by using two operators $\overline{A}^\dagger$ and $\overline{A}$ given by

$$\overline{A} = \overline{a} + \frac{c\sqrt{2}}{Q} \qquad \overline{A}^\dagger = \overline{a}^\dagger + \frac{c\sqrt{2}}{Q} \tag{3.20}$$

where c is a constant such that $\lambda = c(c+1)$. The operator $\overline{T}_0$ is then given by

$$\overline{T}_0 = \tfrac{1}{2}\overline{A}^\dagger\overline{A} + \tfrac{1}{4} - \tfrac{1}{2}c. \tag{3.21}$$

This may play a role in connection with supersymmetry (Witten 1981).

4. The algebraic model

We now come to the central problem of this work. The algebraic model is written as (1.2) with all the above-mentioned restrictions. We suppose that the algebraic model can be solved using group theory. As equation (1.2) must be equivalent to a second-order differential equation (1.1), a sufficient condition would be to take a linear function h when the generators of $SL(2, R)$ are quadratic in P and a quadratic function h when they are linear in P. This will be discussed in more details in the following subsections.

4.1. Linear generators

This case has already been discussed elsewhere (Turbiner 1988) from another point of view. For the function h in (1.2) we take the most general quadratic function of the generators T_i

$$h = -\sum_{i \leq j} a_{ij} T_i T_j + \sum_i b_i T_i \tag{4.1}$$

where i and j take the values $+$, 0 and $-$. When we replace the generators T_i by their expression (3.7) we get

$$h = u(Q)P^2 + v(Q)P + w(Q) \tag{4.2}$$

with

$$\begin{aligned} u(Q) &= -\left(\frac{f(Q)}{f'(Q)}\right)^2 A(Q) \\ v(Q) &= \mathrm{i}\frac{f(Q)}{f'(Q)}\left[\left(1 - \frac{f(Q)f''(Q)}{[f'(Q)]^2} + 2F(Q)\right)A(Q) + B(Q)\right] \\ w(Q) &= \left(\frac{f(Q)F'(Q)}{f'(Q)} + F^2(Q)\right)A(Q) + F(Q)B(Q) + C(Q) \end{aligned} \tag{4.3}$$

and where we use the notation

$$\begin{aligned}
A(Q) &= a_{++}f^2(Q) + a_{+0}f(Q) + a_{+-} + a_{00} + a_{0-}f^{-1}(Q) + a_{--}f^{-2}(Q) \\
B(Q) &= a_{++}(1-2\lambda)f^2(Q) - a_{+0}\lambda f(Q) - a_{+-} - a_{0-}(1-\lambda)f^{-1}(Q) \\
&\qquad - a_{--}(1-2\lambda)f^{-2}(Q) + b_+ f(Q) + b_0 + b_- f^{-1}(Q) \\
C(Q) &= -a_{++}\lambda(1-\lambda)f^2(Q) - a_{+-}\lambda(1+\lambda) - a_{0-}\lambda f^{-1}(Q) \\
&\qquad - a_{--}\lambda(1-\lambda)f^{-2}(Q) - b_+\lambda f(Q) + b_-\lambda f^{-1}(Q).
\end{aligned} \tag{4.4}$$

If we want the eigenvalue problem

$$(h-\varepsilon)|\Psi\rangle = 0 \tag{4.5}$$

to become a Schrödinger equation, we certainly have to impose $v(Q) = 0$ which means that we only take those realisations (3.7) of SL$(2, R)$ for which

$$F(Q) = \frac{f(Q)f''(Q)}{2[f'(Q)]^2} - \frac{1}{2} - \frac{B(Q)}{2A(Q)}. \tag{4.6}$$

The eigenvalue problem (4.5) is now written as

$$\left(\frac{1}{2}P^2 + \frac{w(Q)}{2u(Q)} - \frac{\varepsilon}{2u(Q)}\right)|\Psi\rangle = 0 \tag{4.7}$$

where there still remains one arbitrary function $f(Q)$. Two possibilities may occur: $u(Q)$ is constant or it is not. In the first case the eigenvalues of the algebraic problem (4.5) remain eigenvalues for the 'physical problem' (4.7) up to a constant. This leads in general to an elliptic function for $f(Q)$, the solution of the following differential equation:

$$[f'(Q)]^2 = a_{++}f^4(Q) + a_{+0}f^3(Q) + (a_{+-} + a_{00})f^2(Q) + a_{0-}f(Q) + a_{--}. \tag{4.8}$$

A special interesting application will be discussed in the following subsection. In the second case, equation (4.7) must be interpreted as a Schrödinger equation with a potential

$$V(Q) = \frac{w(Q) - \varepsilon}{2u(Q)} \tag{4.9}$$

depending on one parameter ε. This means that, by solving (4.5), we only get the null eigenvalue of the physical problem (4.7), but for a family of potentials. Examples of both cases are found in Turbiner (1988).

4.2. Quadratic generators

The algebraic model function h is here chosen as a general linear function of the generators

$$h = aT_0 + bT_+ + cT_-. \tag{4.10}$$

The replacement of the generators by their expression (3.16) leads again to the form (4.2) for h, but where $u(Q)$, $v(Q)$ and $w(Q)$ are now given by

$$
\begin{aligned}
u(Q) &= \frac{a+b+c}{[f'(Q)]^2} \\
v(Q) &= (a+b+c)\left(2\frac{g(Q)}{f'(Q)} + \mathrm{i}\frac{f''(Q)}{[f'(Q)]^3}\right) + (b-c)\frac{\mathrm{i}}{2}\frac{f(Q)}{f'(Q)} \\
w(Q) &= (a+b+c)\left(g^2(Q) - \mathrm{i}\frac{g'(Q)}{f'(Q)} + \frac{\lambda}{f^2(Q)}\right) \\
&\quad + \frac{1}{16}(a-b-c)f^2(Q) + (b-c)\left(\frac{\mathrm{i}}{2}f(Q)g(Q) + \frac{1}{4}\right).
\end{aligned} \tag{4.11}
$$

As in the linear case, we will impose $v(Q) = 0$. This allows us to express $g(Q)$ as a function of $f(Q)$

$$
g(Q) = -\mathrm{i}\left[\frac{1}{2}\frac{f''(Q)}{[f'(Q)]^2} + \frac{1}{4}\left(\frac{b-c}{a+b+c}\right)f(Q)\right]. \tag{4.12}
$$

This last condition gives for $w(Q)$ the new expression

$$
\begin{aligned}
w(Q) &= \frac{3}{4}(a+b+c)\frac{[f''(Q)]^2}{[f'(Q)]^4} - \frac{1}{2}(a+b+c)\frac{f'''(Q)}{[f'(Q)]^3} \\
&\quad + \lambda(a+b+c)\frac{1}{f^2(Q)} + \frac{1}{16}\left(\frac{(b-c)^2}{a+b+c} + (a-b-c)\right)f^2(Q)
\end{aligned} \tag{4.13}
$$

while $u(Q)$ is not changed as it does not depend on $g(Q)$. We see immediately that if we want a Schrödinger equation with a spectrum given by that of the algebraic model, that is to say if we restrict to the case of constant $u(Q)$, we only get the trivial case corresponding to the harmonic oscillator. We have indeed $f(Q) = Q$ up to a constant and thus

$$
w(Q) = \frac{1}{16}\left(\frac{(b-c)^2}{a+b+c} + (a-b-c)\right)Q^2 + \lambda(a+b+c)Q^{-2}. \tag{4.14}
$$

The only interesting case thus appears to be that where $u(Q)$ is not a constant. The associated potential is then given by

$$
\begin{aligned}
V(Q) &= \frac{w(Q)-\varepsilon}{2u(Q)} \\
&= \frac{3}{8}\left(\frac{f''(Q)}{f'(Q)}\right)^2 - \frac{1}{4}\frac{f'''(Q)}{f'(Q)} + \frac{1-4b}{32}f^2(Q)[f'(Q)]^2 + \frac{\lambda}{2}\frac{[f'(Q)]^2}{f^2(Q)} - \frac{\varepsilon}{2}[f'(Q)]^2
\end{aligned} \tag{4.15}
$$

where we used, without loss of generality, the conditions $a+b+c = 1$ and $b = c$. This potential will be discussed more carefully in section 6.

5. Example 1

The first example we discuss belongs to the linear case and concerns a special one-dimensional anharmonic potential containing terms in Q^4 and Q^6:

$$V(Q) = \tfrac{1}{2}\omega^2 Q^2 + \tfrac{1}{4}\xi Q^4 + \tfrac{1}{6}\eta Q^6. \tag{5.1}$$

Flessas (1981) showed that one may find exact solutions of the Schrödinger equation associated with this potential if the coefficients fulfil some condition

$$\omega^2 + \sqrt{3\eta} = 3\xi^2/16\eta \tag{5.2}$$

or

$$\omega^2 + 5\sqrt{\eta/3} = 3\xi^2/16\eta \tag{5.3}$$

depending on the parity. We will show now that this peculiar situation has a group theoretical foundation. For this, we have to go back to the general form of the potential in (4.7) and make some particular choice for the parameters. We also make the assumption that the irreducible representation (IR) we are concerned with is the one-dimensional representation $\lambda = 0$. In that case we are sure that the problem is exactly solvable because T_0 has only one eigenvector $|\Psi_0\rangle$ with eigenvalue zero and furthermore the action of the other generators is also equal to zero

$$T_0|\Psi_0\rangle = T_+|\Psi_0\rangle = T_-|\Psi_0\rangle = 0. \tag{5.4}$$

From this we deduce that whatever the choice we make for the parameters a_{ij} and b_i in (4.1), the eigenvalue ε of (4.5) is equal to zero in this IR. In order to get the potential (5.1), we assume that all the coefficients a_{ij} are equal to zero except $a_{+0} = a$ and that the arbitrary function $f(Q)$ is proportional to $1/Q^2$. A very simple calculation gives for $u(Q)$ and $w(Q)$ defined by (4.3), the following expressions:

$$\begin{aligned} u(Q) &= -\frac{a}{4} \\ w(Q) &= \frac{b_0}{2} - \frac{b_+b_0}{2a} + \frac{a}{16Q^2} - \frac{b_+^2}{4aQ^2} + b_-Q^2 - \frac{b_+b_-}{2a}Q^2 - \frac{b_0^2}{4a}Q^2 - \frac{b_0b_-}{2a}Q^4 - \frac{b_-^2}{4a}Q^6 \end{aligned} \tag{5.5}$$

and for the potential (4.9)

$$\begin{aligned} V(Q) = &-\frac{b_0}{a} + \frac{b_+b_0}{a^2} - \frac{1}{8Q^2} + \frac{b_+^2}{2a^2Q^2} \\ &- \frac{2b_-}{a}Q^2 + \frac{b_+b_-}{a^2}Q^2 + \frac{b_0^2}{2a^2}Q^2 + \frac{b_0b_-}{a^2}Q^4 + \frac{b_-^2}{2a^2}Q^6. \end{aligned} \tag{5.6}$$

Comparing this last expression with (5.1) we see that they are identical if we impose $b_+ = \pm\tfrac{1}{2}a$. For $b_+ = \tfrac{1}{2}a$, the Schrödinger equation (4.7) becomes

$$(\tfrac{1}{2}P^2 + \tfrac{1}{2}\omega^2Q^2 + \tfrac{1}{4}\xi Q^4 + \tfrac{1}{6}\eta Q^6 - E)|\Psi\rangle = 0 \tag{5.7}$$

where

$$\omega^2 = \frac{b_0^2}{a^2} - \frac{3b_-}{a} \qquad \xi = \frac{4b_0b_-}{a^2} \qquad \eta = \frac{3b_-^2}{a^2} \tag{5.8}$$

and

$$E = \frac{b_0}{2a}. \tag{5.9}$$

It is very easy to verify that ω, η and ξ satisfy the condition (5.2). The realisation of $\mathrm{SL}(2, R)$ related to this problem is given by

$$\begin{aligned}
T_0 &= -\frac{i}{2}QP - \frac{b_0}{2a}Q^2 - \frac{b_-}{2a}Q^4 \\
T_+ &= -\frac{i}{2Q}P - \frac{b_0}{2a} - \frac{b_-}{2a}Q^2 \\
T_- &= -\frac{i}{2}Q^3P - \frac{b_0}{2a}Q^4 - \frac{b_-}{2a}Q^6.
\end{aligned} \tag{5.10}$$

It is also possible to calculate the wavefunction. It is indeed the solution of $T_0|\Psi_0\rangle = 0$. In the Schrödinger picture, this becomes

$$-\frac{1}{2}q\frac{\mathrm{d}\Psi_0(q)}{\mathrm{d}q} - \frac{b_0}{2a}q^2\Psi_0(q) - \frac{b_-}{2a}q^4\Psi_0(q) = 0 \tag{5.11}$$

and thus

$$\Psi_0(q) = K\exp\left(-\frac{b_0}{2a}q^2 - \frac{b_-}{4a}q^4\right). \tag{5.12}$$

When $b_+ = -\frac{1}{2}a$, the analysis is similar. The Schrödinger equation (4.7) takes the same form as (5.7) but with

$$\omega^2 = \frac{b_0^2}{a^2} - \frac{5b_-}{a} \qquad \xi = \frac{4b_0b_-}{a^2} \qquad \eta = \frac{3b_-^2}{a^2} \tag{5.13}$$

and

$$E = \frac{3b_0}{2a}. \tag{5.14}$$

Now ω, η and ξ obey (5.3) and the generators are

$$\begin{aligned}
T_0 &= -\frac{i}{2}QP + \frac{1}{2} - \frac{b_0}{2a}Q^2 - \frac{b_-}{2a}Q^4 \\
T_+ &= -\frac{i}{2Q}P + \frac{1}{2Q^2} - \frac{b_0}{2a} - \frac{b_-}{2a}Q^2 \\
T_- &= -\frac{i}{2}Q^3P + \frac{1}{2}Q^2 - \frac{b_0}{2a}Q^4 - \frac{b_-}{2a}Q^6.
\end{aligned} \tag{5.15}$$

The wavefunction, the solution of a differential equation related to (5.15) and analogous to (5.11) is given by

$$\Psi_0(q) = Kq\exp\left(-\frac{b_0}{2a}q^2 - \frac{b_-}{4a}q^4\right). \tag{5.16}$$

It is clear that other results can be obtained for this kind of potential by looking, for example, to other finite-dimensional IR of $\mathrm{SL}(2, R)$. They correspond to integer or half-integer values of λ and lead to the diagonalisation of the matrix associated with the operator h (Leach 1984, Turbiner and Ushveridze 1987).

6. Example 2

The second example is related to the quadratic generators. We have seen that the algebraic model (4.10) becomes a Schrödinger equation

$$[\tfrac{1}{2}P^2 + V(Q)]|\Psi\rangle = 0 \tag{6.1}$$

where $V(Q)$ is given by (4.15). With the choice of parameters ($a+b+c=1$ and $b=c$) made before, (1.2) can be written as

$$[(1-2b)\,T_0 + b\,(T_+ + T_-) - \varepsilon]|\Psi\rangle = 0 \tag{6.2}$$

where $|\Psi\rangle$ is the same eigenfunction as in (6.1). Let us restrict our attention to the case of a discrete spectrum related to a discrete positive irreducible representation D_k^+ for which the eigenvalues m of T_0 are bound from below :

$$m = k, k+1, k+2, \ldots. \tag{6.3}$$

In that case the Casimir operator (3.8) is given by

$$C = \tfrac{3}{16} - \tfrac{1}{4}\lambda = k(1-k). \tag{6.4}$$

If we denote by $|km\rangle$ the eigenvectors of T_0, we may assume that

$$|\Psi\rangle = \sum_{m=0}^{\infty} c_m |km\rangle. \tag{6.5}$$

Putting this expression of $|\Psi\rangle$ in equation (6.2) leads to a three-term recursion that can be solved exactly as explained by Ojha (1986). Another method is based on tilted states (Wybourne 1974). To do this, we first introduce two operators

$$T_1 = \frac{1}{2}(T_+ + T_-) \qquad T_2 = \frac{1}{2\mathrm{i}}(T_+ - T_-) \tag{6.6}$$

by analogy with what is commonly done with the SU(2) group. Equation (6.2) can now be written as

$$[(1-2b)\,T_0 + 2b\,T_1 - \varepsilon]|\Psi\rangle = 0. \tag{6.7}$$

In order to solve this equation, we now perform a 'rotation' around the 2-axis. The generators T_0 and T_1 become

$$\begin{aligned} \exp(\mathrm{i}\theta T_2)\,T_0\,\exp(-\mathrm{i}\theta T_2) &= \cosh\theta\,T_0 - \sinh\theta\,T_1 \\ \exp(\mathrm{i}\theta T_2)\,T_1\,\exp(-\mathrm{i}\theta T_2) &= \cosh\theta\,T_1 - \sinh\theta\,T_0 \end{aligned} \tag{6.8}$$

and equation (6.7) is transformed into

$$\{[(1-2b)\cosh\theta - 2b\sinh\theta]T_0 + [2b\cosh\theta - (1-2b)\sinh\theta]T_1\}|\widetilde{\Psi}\rangle = \varepsilon|\widetilde{\Psi}\rangle \tag{6.9}$$

where $|\widetilde{\Psi}\rangle$ is given by

$$|\widetilde{\Psi}\rangle = \exp(\mathrm{i}\theta T_2)|\Psi\rangle. \tag{6.10}$$

If we now choose θ so that the coefficient of T_1 vanishes, we have

$$\tanh\theta = \frac{2b}{1-2b} \tag{6.11}$$

and the equation (6.9) becomes

$$\sqrt{1-4b}\, T_0|\widetilde{\Psi}\rangle = \varepsilon|\widetilde{\Psi}\rangle \tag{6.12}$$

from which we deduce

$$\frac{\varepsilon}{\sqrt{1-4b}} = m = k+n \qquad n = 0, 1, 2, \ldots. \tag{6.13}$$

Let us come back to the physical problem (6.1). We see that this equation depends not only on the eigenvalue ε of the algebraic model, but also on two parameters b and λ, the last fixing the irreducible representation. We have already mentioned the case where ε is the eigenvalue of the physical problem. This corresponds to the harmonic oscillator. The role of the eigenvalue can also be played by b or λ. In all cases, this amounts to writing (4.15) as

$$V(Q) = \overline{V}(Q) - E. \tag{6.14}$$

In the case where the role of the eigenvalue E is played by the term containing b, this means that $f(Q)f'(Q)$ has to be a constant. In order to simplify the expressions, we choose the arbitrary constants of integration in such a way that $f(Q) = \sqrt{Q}$ (to avoid negative values, we may assume that Q is related to the radial coordinate of a three-dimensional problem). Replacing this expression of $f(Q)$ and the corresponding $g(Q)$, given by (4.12), in the general expression for the generators (3.16), we get

$$T_0 = 4QP^2 + \frac{Q}{16} + \frac{\lambda - 3/4}{Q} \qquad T_1 = 4QP^2 - \frac{Q}{16} + \frac{\lambda - 3/4}{Q} \qquad T_2 = QP. \tag{6.15}$$

With this realisation of SL$(2, R)$, equation (6.7) leads to

$$\left(\frac{1}{2}P^2 - \frac{\varepsilon}{8Q} + \frac{\lambda - 3/4}{8Q^2} + \frac{1-4b}{128}\right)|\Psi\rangle = 0. \tag{6.16}$$

This is the radial Schrödinger equation of the attractive Coulomb potential if $\Psi(q)$ represents the radial wavefunction multiplied by q, $\varepsilon/8$ is the charge of the particle, $E = -(1-4b)/128$ is the energy and $k^2 - k = \frac{1}{4}\lambda - \frac{3}{16} = l(l+1)$ is the centrifugal force. This last condition gives two values for k, $k = -l$ and $k = l+1$. We only keep $k = l+1$ that corresponds to bound states. The quantisation (6.13) yields

$$E = -\frac{(\varepsilon/8)^2}{2\,(n+l+1)^2}. \tag{6.17}$$

In the case where the role of the eigenvalue E is played by λ, this implies that $f(Q)/f'(Q)$ is a constant. Among the solutions we choose $f(Q) = \sqrt{8}\, e^{-Q/2}$. The generators of SL(2, R) are then given by

$$T_0 = \frac{e^Q}{2}P^2 + \frac{1+4\lambda}{32}e^Q + \frac{e^{-Q}}{2} \qquad T_1 = \frac{e^Q}{2}P^2 + \frac{1+4\lambda}{32}e^Q - \frac{1}{2}e^{-Q} \qquad T_2 = -P - \frac{i}{2} \tag{6.18}$$

and equation (6.7) becomes

$$\left(\frac{1}{2}P^2 + \frac{1+4\lambda}{32} + \frac{1-4b}{2}\, e^{-2Q} - \varepsilon\, e^{-Q}\right)|\Psi\rangle = 0. \tag{6.19}$$

This is the one-dimensional Morse potential (Berrondo and Palma 1980) if we define $(1-4b)/2 = D$, $\varepsilon = 2D$ and $E = -\frac{1}{32}(1+4\lambda) = -\frac{1}{8} - \frac{1}{2}k^2 + \frac{1}{2}k$. The quantisation (6.13) now yields

$$\sqrt{2D} = k + n \tag{6.20}$$

which gives, if D is fixed, the possible irreducible representations D_k^+. From this last relation we obtain the finite number of bound states

$$E = -D + \sqrt{2D}(n + \tfrac{1}{2}) - \tfrac{1}{2}(n + \tfrac{1}{2})^2. \tag{6.21}$$

This completes the analysis and shows that the harmonic oscillator, the Coulomb potential and the Morse potential are three well known exactly solvable problems connected with the same type of realisations of SL(2, R).

7. Conclusions

In this paper we have analysed realisations of SL(2, R) where the generators are linear or quadratic functions of P, and we have applied this to some algebraic models. It should be very interesting to generalise this analysis to higher polynomials in P. The extension to the n-dimensional case should also be interesting to investigate. This is particularly adapted to the factorisation method (Infeld and Hull 1951) when we consider its group theoretical extension by adding an extra variable as proposed by Kaufman (1965). The analysis presented here can also be considered as a study of some boson representations of SL(2, R). It is indeed obvious to modify the results if we look for realisations of the form

$$T_i = \alpha_i(a^\dagger)\, a + \beta_i(a^\dagger) \tag{7.1}$$

or

$$T_i = \alpha_i(a)\, a^\dagger + \beta_i(a) \tag{7.2}$$

where a and $a^\dagger$ are usual boson operators. The recent results of Doebner *et al* (1989) belong to this class of boson representations.

Acknowledgments

We would like to thank G Reidemeister for very useful discussions on the subject.

References

Berrondo M and Palma A 1980 *J. Phys. A: Math. Gen.* **13** 773
Doebner H D, Gruber B and Lorente M 1989 *J. Math. Phys.* **30** 594
Flessas G P 1981 *Phys. Lett.* **81A** 17
Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
Kaufman B 1965 *J. Math. Phys.* **7** 447
Lánik J 1967 *Nucl. Phys.* B **2** 263
Leach P L G 1984 *J. Math. Phys.* **25** 2974
Mello P A and Moshinsky M 1975 *J. Math. Phys.* **16** 2017
Miller W 1968 *Lie Theory and Special Functions* (New York: Academic)
Ojha P C 1986 *Phys. Rev.* A **34** 969
Sukumar C V 1986 *J. Phys. A: Math. Gen.* **19** 2229
Turbiner A V 1988 *Commun. Math. Phys.* **118** 467
Turbiner A V and Ushveridze A G 1987 *Phys. Lett.* **126A** 181
Witten E 1981 *Nucl. Phys.* B **188** 513
Wybourne B G 1974 *Classical Groups for Physicists* (New York: Wiley)